Minimising Dirichlet eigenvalues on cuboids of unit measure

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Abstract

We consider the minimisation of Dirichlet eigenvalues λ_k, $k \in \mathbb{N}$, of the Laplacian on cuboids of unit measure in $\mathbb{R}^3$. We prove that any sequence of optimal cuboids in $\mathbb{R}^3$ converges to a cube of unit measure in the sense of Hausdorff as $k \to \infty$. We also obtain an upper bound for that rate of convergence.

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1 Introduction.

The eigenvalues of the Laplacian have been the object of intensive study over the last century. Of particular interest are related shape optimisation problems. For $k \in \mathbb{N}$, the goal is to optimise the k'th eigenvalue of the Laplacian with boundary conditions over a collection of open sets in $\mathbb{R}^m$. This collection satisfies geometric constraints, such as fixed Lebesgue measure or fixed perimeter.

For an open set $\Omega \subset \mathbb{R}^m$, $m \geq 2$, of finite Lebesgue measure $|\Omega|$, we let $\lambda_k(\Omega)$, $k \in \mathbb{N}$, denote the Dirichlet eigenvalues of the Laplacian on Ω which are strictly positive, arranged in non-decreasing order and counted with multiplicity:

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \dots \leq \lambda_k(\Omega) \leq \dots$$

This sequence accumulates at $+\infty$.

We consider the following minimisation problem:

$$\lambda_k^*(m) := \inf\{\lambda_k(\Omega) : \Omega \text{ open in } \mathbb{R}^m,\ |\Omega| = c\}.$$

It was shown by Faber and Krahn that among all open sets in $\mathbb{R}^m$ of measure c, the ball of measure c minimises the first Dirichlet eigenvalue, see [15]. Krahn and Szegö proved that, among all open sets in $\mathbb{R}^m$ of measure c, the second Dirichlet eigenvalue is minimised by the union of two disjoint balls of measure $\frac{c}{2}$ each, see [15]. For $k \geq 3$, the existence of an open set of prescribed measure which minimises the k'th Dirichlet eigenvalue remains unresolved to date. However, in the class of quasi-open sets of prescribed measure, it was shown by Bucur in [7] that a minimiser does exist and that such a minimiser is bounded and has finite perimeter. Independently, Mazzoleni and Pratelli proved the existence of a minimiser in [17] in the collection of quasi-open sets. For any lower semi-continuous,

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increasing function of the first k Dirichlet eigenvalues, they proved the existence of a minimiser which is bounded in terms of k and m independently of the function. It was shown in [5] that for $k \le m+1$, any bounded minimiser of $\lambda_k(\Omega)$ has at most $\min\{7,k\}$ components.

No optimal domains are known for λ_k with $k \ge 3$. In particular, the conjecture that if $m=2$, then $\lambda_3(\Omega)$ is bounded from below by the third eigenvalue of the disc with the same measure as Ω is open. There are no obvious candidates for minimisers of λ_k with $k \ge 5$ in any dimension $m \ge 2$. Even for $m=2$, minimisers need not be discs or disjoint unions of discs, see [21]. Furthermore, it was shown in [6] that for $k \ge 5$, $\lambda_k(\Omega)$ cannot be minimised by a disc or a disjoint union of discs. The numerical investigation [1] suggests that for some values of k the minimisers may not have any symmetries.

Pólya's conjecture for Dirichlet eigenvalues asserts that for all bounded, open sets $\Omega \subset \mathbb{R}^m$, $\lambda_k(\Omega) \ge 4\pi^2(\omega_m|\Omega|)^{-2/m}k^{2/m}$, where ω_m denotes the measure of a ball in $\mathbb{R}^m$ of radius 1. It was shown in [11] that Pólya's conjecture is equivalent to $\lambda_k^*(m)$ being asymptotically equal to $4\pi^2(\omega_m c)^{-2/m}k^{2/m}$ as $k \to \infty$.

It is also interesting to consider the optimisation of the eigenvalues of the Laplacian subject to other geometric constraints, such as fixed perimeter. For the Dirichlet eigenvalues, existence of a minimiser in the class of open sets in $\mathbb{R}^m$ of finite Lebesgue measure and prescribed perimeter was shown in [12]. Moreover, it was shown there that any minimiser is bounded and connected, and regularity results for the boundary were also obtained. Bucur and Freitas, [9], showed that any sequence of minimisers of λ_k in $\mathbb{R}^2$ with perimeter ℓ converges in the sense of Hausdorff to the disc of perimeter ℓ as $k \to \infty$. They also showed that if the collection of admissible sets is restricted to the collection of n-sided, convex, planar polygons of perimeter ℓ, then any sequence of minimisers converges to the regular n-sided polygon of perimeter ℓ as $k \to \infty$. For $m \ge 2$, other constraints were considered in [4], including perimeter and moment of inertia, subject to an additional convexity constraint. Further results for the Dirichlet eigenvalues were obtained in [3], [8], [5], [9] and [4]. Some of the results of [3] follow directly from those in [4], while the results of [12] supersede those of [8].

Recently, Antunes and Freitas considered the problem of minimising λ_k over all planar rectangles of unit measure, [2]. In Theorem 2.1 of [2], they showed that any sequence of minimising rectangles for the Dirichlet eigenvalues converges to the unit square in the sense of Hausdorff as $k \to \infty$.

In Theorem 1.1 below we obtain the corresponding 3-dimensional result for the Dirichlet eigenvalues of the Laplacian on cuboids in $\mathbb{R}^3$ of unit measure. In addition we obtain an estimate for the rate of convergence. Let R_{a_1,a_2,a_3} denote a cuboid in $\mathbb{R}^3$ of side-lengths a_1, a_2, a_3 such that $a_1a_2a_3 = 1$ and $a_1 \le a_2 \le a_3$,

$$R_{a_1,a_2,a_3} = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : 0 < x_1 < a_1, 0 < x_2 < a_2, 0 < x_3 < (a_1a_2)^{-1}, a_1 \le a_2 \le a_3\}. \tag{1.1}$$

We prove the following.

Theorem 1.1 (i) *Let $k \in \mathbb{N}$. The variational problem*

$$\lambda_k^* := \inf\{\lambda_k(R_{a_1,a_2,a_3}) : a_1 \le a_2 \le a_3\}$$

has a minimising cuboid $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}$ with side-lengths $a_{1,k}^* \le a_{2,k}^* \le a_{3,k}^*$, such that $a_{1,k}^*a_{2,k}^*a_{3,k}^* = 1$.

(ii)

$$a_{3,k}^* \le 1 + O(k^{-(2-\beta)/6}),\ k \to \infty, \tag{1.2}$$

where β is an exponent of the remainder in

$$\#\{(i_1,i_2,i_3) \in \mathbb{Z}^3 : i_1^2 + i_2^2 + i_3^2 \le R^2\} - \frac{4\pi}{3}R^3 = O(R^\beta)\,, R \to \infty.$$

Furthermore, any sequence of optimal cuboids $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}$ converges to the unit cube in $\mathbb{R}^3$ in the sense of Hausdorff as $k \to \infty$.

The best known estimate to date is that for any $\epsilon > 0$, $\beta = \frac{21}{16} + \epsilon$, see [14]. Hence (1.2) holds for $\beta = \frac{21}{16} + \epsilon, \epsilon > 0$. The conjecture for the optimal remainder is $\beta = 1 + \epsilon, \epsilon > 0$. See [10].

A heuristic explanation for this asymptotic shape result is the following (see also [2]). For any cuboid R in $\mathbb{R}^3$ with measure $|R|$ and perimeter $\mathrm{Per}(R)$, one has that

$$\lambda_k(R) = \left(\frac{6\pi^2 k}{|R|}\right)^{2/3} + \frac{(3\pi^5)^{1/3}\mathrm{Per}(R)k^{1/3}}{2^{5/3}|R|^{4/3}} + o(k^{1/3}),\ k \to \infty. \tag{1.3}$$

So if $|R| = 1$ then (1.3) suggests that the cuboid that minimises $\lambda_k(R),\ k \to \infty$, is the one with minimal perimeter, i.e. the unit cube.

The Dirichlet eigenvalues of the Laplacian on a cuboid R_{a_1,a_2,a_3} (as in (1.1)) are given by

$$\frac{\pi^2 i_1^2}{a_1^2} + \frac{\pi^2 i_2^2}{a_2^2} + \frac{\pi^2 i_3^2}{a_3^2}, \quad i_1, i_2, i_3 \in \mathbb{N}. \tag{1.4}$$

By listing these in non-decreasing order including multiplicities, the k'th Dirichlet eigenvalue on R_{a_1,a_2,a_3}, $\lambda_k(R_{a_1,a_2,a_3})$, is the k'th item of this list. In the table below we list the minimising cuboids for the first few Dirichlet eigenvalues.

k	λ_k^*	$a_{1,k}^*, a_{2,k}^*, a_{3,k}^*$	Minimising modes
1	$3\pi^2$	$1, 1, 1$	$(1,1,1)$
2	$3 \cdot 2^{2/3}\pi^2$	$2^{-1/3}, 2^{-1/3}, 2^{2/3}$	$(1,1,2)$
3	$3 \cdot 2^{-2/3}5^{2/3}\pi^2$	$\left(\frac{2}{5}\right)^{1/3}, \left(\frac{5}{2}\right)^{1/6}, \left(\frac{5}{2}\right)^{1/6}$	$(1,2,1)$
4	$6\pi^2$	$1, 1, 1$	$(2,1,1)$
5	$3^{5/3}\pi^2$	$3^{-1/3}, 3^{-1/3}, 3^{2/3}$	$(1,1,3)$
6	$3 \cdot 2^{4/3}\pi^2$	$2^{-2/3}, 2^{1/3}, 2^{1/3}$ or $2^{-2/3}, 2^{-2/3}, 2^{4/3}$	$(1,2,2)$ or $(1,1,4)$
7	$3 \cdot 5^{2/3}\pi^2$	$\left(\frac{5}{8}\right)^{1/6}, \left(\frac{5}{8}\right)^{1/6}, 2 \cdot 5^{-1/3}$ or $5^{-1/3}, 5^{-1/3}, 5^{2/3}$	$(2,1,2)$ or $(1,1,5)$
8	$9\pi^2$	$1, 1, 1$	$(2,2,1)$

Let $\lambda \in \mathbb{R}$, $\lambda \geq 0$, and $a_1, a_2, a_3 \in \mathbb{R}$ such that $a_1a_2a_3 = 1$ and $a_1 \leq a_2 \leq a_3$. With (1.4) in mind, we define

$$E(\lambda) := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \leq \frac{\lambda}{\pi^2}\right\}. \tag{1.5}$$

The ellipsoid $E(\lambda)$ has semi-axes

$$r_1 = \frac{a_1\lambda^{1/2}}{\pi}, \quad r_2 = \frac{a_2\lambda^{1/2}}{\pi}, \quad r_3 = \frac{a_3\lambda^{1/2}}{\pi},$$

and $|E(\lambda)| = \frac{4}{3\pi^2}\lambda^{3/2}$.

By (1.4) and (1.5), we see that the Dirichlet eigenvalues $\lambda_1(R_{a_1,a_2,a_3}), \ldots, \lambda_k(R_{a_1,a_2,a_3})$ (counted with multiplicities) correspond to the integer lattice points that are inside or on the ellipsoid $E(\lambda_k)$ in the first octant (excluding the coordinate planes). Thus, in order to minimise λ_k among all cuboids given by (1.1), we wish to determine the 3-dimensional ellipsoid $E(\lambda) \subset \mathbb{R}^3$ of minimal measure which encloses k integer lattice points in the first octant (excluding the coordinate planes).

For $n \in \mathbb{N}$, $n \geq 2$, estimates for the number of integer lattice points which are inside or on an n-dimensional ellipsoid have been widely studied from a number theoretical viewpoint. However, in order to use these estimates, it is crucial that the corresponding cuboids are bounded as $k \to \infty$. As in the 2-dimensional case, this is the most difficult part of the proof.

This paper is organised as follows. In Section 2 we prove Theorem 1.1(i). In Section 3 we obtain bounds for lattice point sums which are key ingredients in the proofs of the lemmas in Section 4. In that section we follow the strategy of [2], and prove that the side-lengths of a sequence of minimal cuboids $\left(R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}\right)_k$ are bounded uniformly in k. This is achieved by first obtaining an upper bound for the counting function $N(\lambda) = \#\{j \in \mathbb{N} : \lambda_j(R_{a_1,a_2,a_3}) \leq \lambda\}$ for arbitrary cuboids. Using the maximality of $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}$, and comparing with the unit cube gives the required uniform bound. Finally in Section 5 we use known estimates for the number of integer lattice points that are inside and on an ellipsoid to conclude the proof of Theorem 1.1(ii).

2 Proof of Theorem 1.1(i).

Proof. Fix $k \in \mathbb{N}$. Suppose that $\{R_{a_{1,k}^{(\ell)},a_{2,k}^{(\ell)},a_{3,k}^{(\ell)}}\}_{\ell\in\mathbb{N}}$ is a minimising sequence for λ_k such that $a_{3,k}^{(\ell)} \to \infty$ as $\ell \to \infty$. In order to preserve the measure constraint $a_{1,k}^{(\ell)} \to 0$ as $\ell \to \infty$. So, we have that

$$\lambda_k\big(R_{a_{1,k}^{(\ell)},a_{2,k}^{(\ell)},a_{3,k}^{(\ell)}}\big) > \frac{\pi^2}{(a_{1,k}^{(\ell)})^2} \to \infty, \text{ as } \ell \to \infty.$$

However, for the unit cube in $\mathbb{R}^3$, $\lambda_k \le 3\pi^2 k^2 < +\infty$. This contradicts the assumption that $\{R_{a_{1,k}^{(\ell)},a_{2,k}^{(\ell)},a_{3,k}^{(\ell)}}\}_{\ell\in\mathbb{N}}$ is a minimising sequence for λ_k. So any minimising sequence $\{R_{a_{1,k}^{(\ell)},a_{2,k}^{(\ell)},a_{3,k}^{(\ell)}}\}_{\ell\in\mathbb{N}}$ for λ_k is such that $a_{1,k}^{(\ell)}, a_{2,k}^{(\ell)}, a_{3,k}^{(\ell)}$ are bounded as $\ell \to \infty$. Hence, for each $i \in \{1,2,3\}$, there exists a convergent subsequence, again denoted by $a_{i,k}^{(\ell)}$ such that $a_{i,k}^{(\ell)} \to a_{i,k}^*$ for some $a_{i,k}^* \in (0,\infty)$. Since $(a_1,a_2,a_3) \mapsto \lambda_k(R_{a_1,a_2,a_3})$ is continuous, $\lambda_k\big(R_{a_{1,k}^{(\ell)},a_{2,k}^{(\ell)},a_{3,k}^{(\ell)}}\big) \to \lambda_k\big(R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}\big)$ as $\ell \to \infty$. Hence $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}$ is a minimising cuboid for λ_k. ∎

It is not difficult to see that the above argument can also be used to prove the existence of a minimising cuboid for λ_k in $\mathbb{R}^m$ with $m \ge 4$.

3 Key lemmas to prove boundedness of an optimal cuboid.

The following lemmas are crucial in the proofs that follow in Section 4.

Lemma 3.1 *Let $y \ge 0$, $a \ge 0$. For $n \in \{1,2\}$, we have that*

$$\sum_{i=1}^{\left\lfloor \frac{y^{1/2}}{a} \right\rfloor} (y - a^2 i^2)^{n/2} \le \frac{\sqrt{\pi}}{2a} \frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)} y^{(n+1)/2} - \frac{1}{2} y^{n/2} + \frac{(2an)^{n/2}}{(n+2)^{(n+2)/2}} y^{n/4}. \tag{3.1}$$

Proof. We have that

$$\sum_{i=1}^{\left\lfloor \frac{y^{1/2}}{a} \right\rfloor} (y - a^2 i^2)^{n/2} = a^n \sum_{i=1}^{\left\lfloor \frac{y^{1/2}}{a} \right\rfloor} \left(\left(\frac{y^{1/2}}{a} \right)^2 - i^2 \right)^{n/2}. \tag{3.2}$$

Let $R = \frac{y^{1/2}}{a}$ and consider $\sum_{i=1}^{\lfloor R \rfloor} g(i)$ where

$$g(i) = (R^2 - i^2)^{n/2}. \tag{3.3}$$

Then, for $0 \le i \le R$, we have that

$$g'(i) = -ni(R^2 - i^2)^{(n-2)/2} \le 0,$$
$$g''(i) = n(R^2 - i^2)^{(n-4)/2}((n-1)i^2 - R^2) \le 0.$$

So $i \mapsto g(i)$ is decreasing on $[0,R]$ and, since $n = 1$ or $n = 2$, g is also concave on $[0,R]$. We note that since g is decreasing, $\sum_{i=1}^{\lfloor R \rfloor} g(i)$ is the total area of the rectangles of width 1 and height $g(i)$, $i \in \{1,\dots,\lfloor R \rfloor\}$, which are inscribed in the curve $g(x)$ for $0 \le x \le R$. Due to the concavity of g on $(0,R)$, we can bound $\sum_{i=1}^{\lfloor R \rfloor} g(i)$ from above by the area under g minus the area of the inscribed triangles which sit on top of the aforementioned rectangles. That is

$$\sum_{i=1}^{\lfloor R \rfloor} g(i) \le \int_0^R g(i)\, di - \frac{1}{2} \sum_{i=1}^{\lfloor R \rfloor} (g(i-1) - g(i)) - \frac{1}{2} \left(R - \lfloor R \rfloor\right) g\left(\lfloor R \rfloor\right). \tag{3.4}$$

We have that

$$\int_0^R g(i)\,di = R^{n+1}\int_0^1 (1-t^2)^{n/2}\,dt$$
$$= \frac{R^{n+1}}{2}\int_0^1 (1-s)^{n/2}\frac{1}{\sqrt{s}}\,ds = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)}R^{n+1}, \tag{3.5}$$

where we have used [3.191.3, 8.384.1, [13]].

We also have that

$$\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{\lfloor R\rfloor}(g(i-1)-g(i)) - \frac{1}{2}\left(R-\lfloor R\rfloor\right)g\left(\lfloor R\rfloor\right)\\
&= -\frac{1}{2}R^n + \frac{1}{2}(1+\lfloor R\rfloor - R)\big(R^2-\lfloor R\rfloor^2\big)^{n/2}\\
&= -\frac{1}{2}R^n + \frac{1}{2}(1+\lfloor R\rfloor - R)(R+\lfloor R\rfloor)^{n/2}(R-\lfloor R\rfloor)^{n/2}\\
&\le -\frac{1}{2}R^n + \frac{1}{2}(2R)^{n/2}\max_{0\le\beta<1}(1-\beta)\beta^{n/2}\\
&= -\frac{1}{2}R^n + \frac{(2n)^{n/2}}{(n+2)^{(n+2)/2}}R^{n/2}.
\end{aligned} \tag{3.6}$$

Combining (3.2), (3.4), (3.5) and (3.6) gives (3.1). ∎

Applying the previous lemma with $n=1$, $y=\frac{a_2^2}{\pi^2}\lambda$, and $a=\frac{a_2}{a_1}$, we recover the result of Theorem 3.1 from [2]. Since g (as in (3.3)) is decreasing on $[0,\frac{y^{1/2}}{a}]$, the following holds for all $n\in\mathbb{N}$.

Lemma 3.2 *Let $y\ge 0$, $a\ge 0$. For $n\in\mathbb{N}$, we have that*

$$\sum_{i=1}^{\left\lfloor \frac{y^{1/2}}{a}\right\rfloor}(y-a^2i^2)^{n/2} \le \int_0^{\frac{y^{1/2}}{a}}(y-a^2i^2)^{n/2}\,di = \frac{\sqrt{\pi}}{2a}\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)}y^{(n+1)/2}.$$

4 Uniform boundedness of an optimal cuboid.

With $E(\lambda)$ as defined in (1.5), we define the counting function

$$N(\lambda) := \#\{j\in\mathbb{N} : \lambda_j(R_{a_1,a_2,a_3})\le\lambda\} = \#\{(i_1,i_2,i_3)\in\mathbb{N}^3\cap E(\lambda)\}.$$

We now use the results of Section 3 to obtain an upper bound for $N(\lambda)$.

Lemma 4.1 *For $\lambda\ge 0$ and $a_1\le a_2\le a_3$, $E(\lambda)$, $N(\lambda)$ as above, we have that*

$$N(\lambda)\le\frac{\lambda^{3/2}}{6\pi^2}-\frac{\lambda}{8\pi a_1}+\frac{\lambda^{1/2}}{16a_1^2}. \tag{4.1}$$

Proof. For $(i_1,i_2,i_3)\in\mathbb{N}^3\cap E(\lambda)$, we have that

$$i_3\le\left\lfloor\left(\frac{a_3^2}{\pi^2}\lambda-\frac{a_3^2}{a_1^2}i_1^2-\frac{a_3^2}{a_2^2}i_2^2\right)_+^{1/2}\right\rfloor,$$

where "+" denotes the positive part. Hence

$$N(\lambda) \leq \sum_{i_1\in\mathbb{N}}\sum_{i_2\in\mathbb{N}} \left\lfloor \left(\frac{a_3^2}{\pi^2}\lambda - \frac{a_3^2}{a_1^2}i_1^2 - \frac{a_3^2}{a_2^2}i_2^2\right)_+^{1/2} \right\rfloor \tag{4.2}$$

$$\leq \sum_{i_1=1}^{\left\lfloor a_1\frac{\lambda^{1/2}}{\pi}\right\rfloor} \sum_{i_2=1}^{\left\lfloor a_2\left(\frac{\lambda}{\pi^2}-\frac{i_1^2}{a_1^2}\right)^{1/2}\right\rfloor} \left(\frac{a_3^2}{\pi^2}\lambda - \frac{a_3^2}{a_1^2}i_1^2 - \frac{a_3^2}{a_2^2}i_2^2\right)^{1/2}. \tag{4.3}$$

Applying Lemma 3.2 with $y = \frac{a_3^2}{\pi^2}\lambda - \frac{a_3^2}{a_1^2}i_1^2$, $a = \frac{a_3}{a_2}$, $n = 1$ to (4.3), we have that

$$N(\lambda) \leq \sum_{i_1=1}^{\left\lfloor a_1\frac{\lambda^{1/2}}{\pi}\right\rfloor} \frac{\pi a_2}{4a_3}\left(\frac{a_3^2}{\pi^2}\lambda - \frac{a_3^2}{a_1^2}i_1^2\right) = \sum_{i_1=1}^{\left\lfloor a_1\frac{\lambda^{1/2}}{\pi}\right\rfloor} \frac{\pi a_2 a_3}{4}\left(\frac{\lambda}{\pi^2} - \frac{i_1^2}{a_1^2}\right). \tag{4.4}$$

Applying Lemma 3.1 with $y = \frac{\lambda}{\pi^2}$, $a = \frac{1}{a_1}$, $n = 2$, we obtain that

$$\frac{\pi a_2 a_3}{4}\sum_{i_1=1}^{\left\lfloor a_1\frac{\lambda^{1/2}}{\pi}\right\rfloor}\left(\frac{\lambda}{\pi^2} - \frac{i_1^2}{a_1^2}\right) \leq \frac{\pi a_2 a_3}{4}\left(\frac{2a_1}{3\pi^3}\lambda^{3/2} - \frac{1}{2\pi^2}\lambda + \frac{1}{4\pi a_1}\lambda^{1/2}\right)$$

$$= \frac{\lambda^{3/2}}{6\pi^2} - \frac{\lambda}{8\pi a_1} + \frac{\lambda^{1/2}}{16a_1^2}. \tag{4.5}$$

By (4.4) and (4.5), (4.1) follows. ∎

We now prove that the side-lengths $a_{1,k}^*, a_{2,k}^*, a_{3,k}^*$, of an optimal cuboid $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}$ in $\mathbb{R}^3$ are uniformly bounded.

Lemma 4.2 *For all $k \in \mathbb{N}$,*

$$a_{3,k}^* \leq 319.$$

Proof. Since (4.1) holds for all $\lambda \geq 0$ and all cuboids, it holds for $\lambda = \lambda_k^*$ and an optimal cuboid $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}$, so

$$k \leq N(\lambda_k^*) \leq \frac{(\lambda_k^*)^{3/2}}{6\pi^2} - \frac{\lambda_k^*}{8\pi a_{1,k}^*} + \frac{(\lambda_k^*)^{1/2}}{16(a_{1,k}^*)^2},$$

and, by rearranging, we obtain that

$$\frac{(\lambda_k^*)^{3/2} - 6\pi^2 k}{6\pi^2\lambda_k^*} \geq \frac{1}{8\pi a_{1,k}^*} - \frac{(\lambda_k^*)^{-1/2}}{16(a_{1,k}^*)^2}. \tag{4.6}$$

The left-hand side of (4.6) is an increasing function of λ_k^*, so it is bounded from above by $\frac{\nu_k^{3/2}-6\pi^2 k}{6\pi^2\nu_k}$, where ν_k is the kth Dirichlet eigenvalue of the Laplacian on the unit cube in $\mathbb{R}^3$. We obtain a lower bound for the right-hand side of (4.6) by using the fact that

$$\lambda_k^* \geq \lambda_1(R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}) \geq \frac{\pi^2}{(a_{1,k}^*)^2},$$

implies that

$$-\frac{(\lambda_k^*)^{-1/2}}{16(a_{1,k}^*)^2} \geq -\frac{1}{16\pi a_{1,k}^*}. \tag{4.7}$$

Hence, by (4.7), we have that

$$\frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2\nu_k} \geq \frac{1}{16\pi a_{1,k}^*},$$

which implies that,

$$a^*_{1,k} \geq \frac{1}{16\pi} \frac{6\pi^2 \nu_k}{\nu_k^{3/2} - 6\pi^2 k}. \tag{4.8}$$

We now obtain a uniform lower bound for $a^*_{1,k}$. Let ω_3 denote the measure of a ball of radius 1 in $\mathbb{R}^3$. Then, by an estimate of Gauss, we have that

$$N(\nu_k) = \#\left\{(i_1, i_2, i_3) \in \mathbb{N}^3 : i_1^2 + i_2^2 + i_3^2 \leq \frac{\nu_k}{\pi^2}\right\} \geq \frac{\omega_3}{8}\left(\frac{\nu_k^{1/2}}{\pi} - 3^{1/2}\right)_+^3 \geq \frac{\nu_k^{3/2}}{6\pi^2} - \frac{3^{1/2}\nu_k}{2\pi}.$$

Let Θ_k denote the multiplicity of ν_k. Then $N(\nu_k) \leq k + \Theta_k - 1$. In addition, $\Theta_k = \#\{(i_1, i_2, i_3) \in \mathbb{N}^3 : i_1^2 + i_2^2 + i_3^2 = \frac{\nu_k}{\pi^2}\}$ is the number of integer lattice points in the first octant that lie on the sphere in $\mathbb{R}^3$ which is centred at $(0,0,0)$ and has radius $\frac{\nu_k^{1/2}}{\pi}$. By projection onto the plane $i_3 = 0$, each of these lattice points corresponds to an integer lattice point which lies inside or on the circle $\{(i_1, i_2) \in \mathbb{Z}^2 : i_1^2 + i_2^2 = \frac{\nu_k}{\pi^2}\}$ in the first quadrant. The number of integer lattice points which lie inside or on this circle is bounded from above by $\frac{\nu_k}{4\pi}$, i.e. the area inscribed by the circle in the first quadrant. Thus we obtain that

$$\nu_k^{3/2} \leq 6\pi^2 k + 3\pi\nu_k\left(\frac{1}{2} + 3^{1/2}\right). \tag{4.9}$$

Hence by (4.8) and (4.9), we have that

$$a^*_{1,k} \geq \left(8\left(\frac{1}{2} + 3^{1/2}\right)\right)^{-1}. \tag{4.10}$$

Using that $a^*_{1,k} \leq a^*_{2,k} \leq a^*_{3,k}$, $a^*_{1,k}a^*_{2,k}a^*_{3,k} = 1$ and (4.10), we deduce that

$$a^*_{3,k} \leq \frac{1}{(a^*_{1,k})^2} \leq 64\left(\frac{1}{2} + 3^{1/2}\right)^2 \leq 319.$$

∎

The main obstructions to proving a corresponding result to Theorem 1.1(ii) in higher dimensions $m \geq 4$ are the following. Firstly, for $m \geq 4$ the corresponding upper bound for $N(\lambda)$ to (4.2) involves lattice point sums $\sum_{i=1}^{\lfloor R \rfloor} g(i)$ with $g(i)$, R as in (3.3) and $n \geq 3$. For $n \geq 3$, $\frac{y^{1/2}}{a\sqrt{n-1}}$ is an inflection point of g in $(0, \frac{y^{1/2}}{a})$ and so g is not concave on $(0, \frac{y^{1/2}}{a})$. Thus, the above approach cannot be used to obtain an upper bound for the left-hand side of (3.1) when $n \geq 3$. Secondly, the higher-dimensional equivalent of (4.1) will contain more terms in the right-hand side. The leading term in that right-hand side is the Weyl term. However, the lower order terms are bounds which are uniform in a_1, for example. Their usefulness depends on the numerical coefficients which show up. These in turn depend on lower dimensional lattice point sums.

5 Proof of Theorem 1.1(ii).

The minimisers $R_{a^*_{1,k}, a^*_{2,k}, a^*_{3,k}}$ of λ_k need not be unique. From this point onwards, we consider an arbitrary subsequence of minimisers denoted by $\left(R_{a^*_{1,k}, a^*_{2,k}, a^*_{3,k}}\right)_k$.

For $E(\lambda)$ as defined in (1.5), we introduce the following notation.

$$T(\lambda) = \#\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \cap E(\lambda)\},$$
$$T_{x_1}(\lambda) = \#\{(0, x_2, x_3) \in (\{0\} \times \mathbb{Z}^2) \cap E(\lambda)\},$$
$$T^+_{x_1}(\lambda) = \#\{(0, x_2, x_3) \in (\{0\} \times \mathbb{N}^2) \cap E(\lambda)\}.$$

$T(\lambda)$ is the total number of integer lattice points that are inside or on the ellipsoid $E(\lambda)$ in $\mathbb{R}^3$. Similarly $T_{x_1}(\lambda)$ is the number of integer lattice points that are inside or on the ellipse in $\mathbb{R}^2$ which is centred at $(0,0)$ and has semi-axes $\frac{a_2\lambda^{1/2}}{\pi}$, $\frac{a_3\lambda^{1/2}}{\pi}$. $T^+_{x_1}(\lambda)$ is the number of these lattice points that lie in the first quadrant (excluding the axes). $T_{x_2}(\lambda)$, $T^+_{x_2}(\lambda)$ etc. are defined similarly. Thus, we have that

$$\begin{aligned} T(\lambda) &= 8N(\lambda) + 4T^+_{x_1}(\lambda) + 4T^+_{x_2}(\lambda) + 4T^+_{x_3}(\lambda) \\ &\quad + 2\left\lfloor \frac{a_1\lambda^{1/2}}{\pi} \right\rfloor + 2\left\lfloor \frac{a_2\lambda^{1/2}}{\pi} \right\rfloor + 2\left\lfloor \frac{a_3\lambda^{1/2}}{\pi} \right\rfloor + 1, \end{aligned}$$

which implies that

$$\begin{aligned} N(\lambda) &= \frac{1}{8}T(\lambda) - \frac{1}{2}T^+_{x_1}(\lambda) - \frac{1}{2}T^+_{x_2}(\lambda) - \frac{1}{2}T^+_{x_3}(\lambda) \\ &\quad - \frac{1}{4}\left\lfloor \frac{a_1\lambda^{1/2}}{\pi} \right\rfloor - \frac{1}{4}\left\lfloor \frac{a_2\lambda^{1/2}}{\pi} \right\rfloor - \frac{1}{4}\left\lfloor \frac{a_3\lambda^{1/2}}{\pi} \right\rfloor - \frac{1}{8}. \end{aligned}$$

In addition, we have that

$$T_{x_1}(\lambda) = 4T^+_{x_1}(\lambda) + 2\left\lfloor \frac{a_2\lambda^{1/2}}{\pi} \right\rfloor + 2\left\lfloor \frac{a_3\lambda^{1/2}}{\pi} \right\rfloor + 1,$$

which implies that

$$T^+_{x_1}(\lambda) = \frac{1}{4}T_{x_1}(\lambda) - \frac{1}{2}\left\lfloor \frac{a_2\lambda^{1/2}}{\pi} \right\rfloor - \frac{1}{2}\left\lfloor \frac{a_3\lambda^{1/2}}{\pi} \right\rfloor - \frac{1}{4},$$

and similarly for $T^+_{x_2}(\lambda), T^+_{x_3}(\lambda)$. Thus, we obtain

$$\begin{aligned} N(\lambda) &= \frac{1}{8}T(\lambda) - \frac{1}{8}T_{x_1}(\lambda) - \frac{1}{8}T_{x_2}(\lambda) - \frac{1}{8}T_{x_3}(\lambda) \\ &\quad + \frac{1}{4}\left\lfloor \frac{a_1\lambda^{1/2}}{\pi} \right\rfloor + \frac{1}{4}\left\lfloor \frac{a_2\lambda^{1/2}}{\pi} \right\rfloor + \frac{1}{4}\left\lfloor \frac{a_3\lambda^{1/2}}{\pi} \right\rfloor + \frac{1}{4}. \end{aligned} \tag{5.1}$$

Below we use this expression for $N(\lambda)$ in order to prove Theorem 1.1(ii).

Proof of Theorem 1.1(ii). By setting $\lambda = \lambda^*_k$ in (5.1) and considering an optimal cuboid $R_{a^*_{1,k},a^*_{2,k},a^*_{3,k}}$, we have that

$$\begin{aligned} k \le N(\lambda^*_k) &= \frac{1}{8}T(\lambda^*_k) - \frac{1}{8}T_{x_1}(\lambda^*_k) - \frac{1}{8}T_{x_2}(\lambda^*_k) - \frac{1}{8}T_{x_3}(\lambda^*_k) \\ &\quad + \frac{1}{4}\left\lfloor \frac{a^*_{1,k}(\lambda^*_k)^{1/2}}{\pi} \right\rfloor + \frac{1}{4}\left\lfloor \frac{a^*_{2,k}(\lambda^*_k)^{1/2}}{\pi} \right\rfloor + \frac{1}{4}\left\lfloor \frac{a^*_{3,k}(\lambda^*_k)^{1/2}}{\pi} \right\rfloor + \frac{1}{4}. \end{aligned} \tag{5.2}$$

By Lemma 4.2, the $\{a^*_{1,k}, a^*_{2,k}, a^*_{3,k}\}$ are uniformly bounded, so it is possible to make use of known estimates for the number of integer lattice points that are inside or on a 3-dimensional ellipsoid or a 2-dimensional ellipse. In particular there exists $C < \infty$ such that for all $\lambda \ge 0$

$$\frac{4}{3\pi^2}\lambda^{3/2} - C\lambda^{\beta/2} \le T(\lambda) \le \frac{4}{3\pi^2}\lambda^{3/2} + C\lambda^{\beta/2} + 1, \tag{5.3}$$

where β is as defined in the Introduction. Similarly there exists $D < \infty$ such that for all $\lambda \ge 0$

$$\frac{a_2a_3}{\pi}\lambda - D\lambda^{\theta/2} \le T_{x_1}(\lambda) \le \frac{a_2a_3}{\pi}\lambda + D\lambda^{\theta/2} + 1, \tag{5.4}$$

where θ is an exponent of the remainder in Gauss' circle problem

$$\#\{(i_1,i_2) \in \mathbb{Z}^2 : i_1^2 + i_2^2 \le R^2\} - \pi R^2 = O(R^\theta)\,, R \to \infty.$$

The best known estimate to date is $\theta > \frac{131}{208}$, see the Introduction in [16]. Hence the formula above holds for $\theta = \frac{131}{208} + \epsilon$ for any $\epsilon > 0$. The corresponding inequalities to (5.4) also hold for $T_{x_2}(\lambda), T_{x_3}(\lambda)$. Using these inequalities and (5.2), we obtain the following upper bound for $N(\lambda_k^*)$.

$$\begin{aligned} k \le N(\lambda_k^*) \le \frac{(\lambda_k^*)^{3/2}}{6\pi^2} - \frac{1}{8\pi}\left(\frac{1}{a_{1,k}^*} + \frac{1}{a_{2,k}^*} + \frac{1}{a_{3,k}^*}\right)\lambda_k^* + \frac{C}{8}(\lambda_k^*)^{\beta/2} \\ + \frac{1}{4\pi}(a_{1,k}^* + a_{2,k}^* + a_{3,k}^*)(\lambda_k^*)^{1/2} + \frac{3D}{8}(\lambda_k^*)^{\theta/2} + \frac{3}{8}. \end{aligned} \tag{5.5}$$

Rearranging (5.5), we obtain that

$$\begin{aligned} \frac{1}{a_{1,k}^*} + \frac{1}{a_{2,k}^*} + \frac{1}{a_{3,k}^*} \le 8\pi\left(\frac{(\lambda_k^*)^{3/2} - 6\pi^2 k}{6\pi^2\lambda_k^*}\right) + \pi C(\lambda_k^*)^{-(2-\beta)/2} \\ + 2(a_{1,k}^* + a_{2,k}^* + a_{3,k}^*)(\lambda_k^*)^{-1/2} + 3\pi D(\lambda_k^*)^{-(2-\theta)/2} + 3\pi(\lambda_k^*)^{-1}. \end{aligned}$$

Since $\frac{(\lambda_k^*)^{3/2} - 6\pi^2 k}{6\pi^2\lambda_k^*}$ is an increasing function of λ_k^*, we can replace λ_k^* by ν_k, where ν_k is the kth Dirichlet eigenvalue of the Laplacian on the unit cube in $\mathbb{R}^3$. Thus, by Pólya's Inequality $\lambda_k^* \ge (6\pi^2 k)^{2/3}$, ([19, 20]), we obtain

$$\begin{aligned} \frac{1}{a_{1,k}^*} + \frac{1}{a_{2,k}^*} + \frac{1}{a_{3,k}^*} &\le 8\pi\left(\frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2\nu_k}\right) + \pi C(\lambda_k^*)^{-(2-\beta)/2} + 3\pi(\lambda_k^*)^{-1} \\ &\quad + 2(a_{1,k}^* + a_{2,k}^* + a_{3,k}^*)(\lambda_k^*)^{-1/2} + 3\pi D(\lambda_k^*)^{-(2-\theta)/2} \\ &\le 8\pi\left(\frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2\nu_k}\right) + \pi C(6\pi^2)^{-(2-\beta)/3}k^{-(2-\beta)/3} + 3\pi(6\pi^2)^{-2/3}k^{-2/3} \\ &\quad + 2(a_{1,k}^* + a_{2,k}^* + a_{3,k}^*)(6\pi^2)^{-1/3}k^{-1/3} + 3\pi D(6\pi^2)^{-(2-\theta)/3}k^{-(2-\theta)/3} \\ &= 8\pi\left(\frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2\nu_k}\right) + O(k^{-(2-\beta)/3}). \end{aligned} \tag{5.6}$$

To obtain an upper bound for $\frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2\nu_k}$ we proceed as follows. By (5.1) with $\lambda = \nu_k$ we have that

$$N(\nu_k) = \frac{1}{8}T(\nu_k) - \frac{3}{8}T_{x_1}(\nu_k) + \frac{3}{4}\left\lfloor \frac{\nu_k^{1/2}}{\pi} \right\rfloor + \frac{1}{4}. \tag{5.7}$$

Since $a_1 = a_2 = a_3 = 1$, by (5.3) and (5.4), we have that

$$\frac{4}{3\pi^2}\nu_k^{3/2} - C\nu_k^{\beta/2} \le T(\nu_k), \tag{5.8}$$

and

$$T_{x_1}(\nu_k) \le \frac{\nu_k}{\pi} + D\nu_k^{\theta/2} + 1, \tag{5.9}$$

where β and θ are as in (5.3), (5.4). Again let Θ_k denote the multiplicity of ν_k. Thus by (5.7), (5.8) and (5.9), we obtain a lower bound for $N(\nu_k)$:

$$k + \Theta_k - 1 \ge N(\nu_k) \ge \frac{\nu_k^{3/2}}{6\pi^2} - \frac{C}{8}\nu_k^{\beta/2} - \frac{3}{8\pi}\nu_k - \frac{3D}{8}\nu_k^{\theta/2} + \frac{3}{4\pi}\nu_k^{1/2} - \frac{7}{8},$$

which implies that

$$\begin{aligned} \frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2\nu_k} &\le \frac{3}{8\pi} + \frac{C}{8}\nu_k^{-(2-\beta)/2} + \frac{3D}{8}\nu_k^{-(2-\theta)/2} - \frac{3}{4\pi}\nu_k^{-1/2} + \Theta_k\nu_k^{-1} - \frac{1}{8}\nu_k^{-1} \\ &\le \frac{3}{8\pi} + \frac{C}{8}\nu_k^{-(2-\beta)/2} + \frac{3D}{8}\nu_k^{-(2-\theta)/2} + \Theta_k\nu_k^{-1} \\ &\le \frac{3}{8\pi} + \frac{C}{8}(6\pi^2)^{-(2-\beta)/3}k^{-(2-\beta)/3} + \frac{3D}{8}(6\pi^2)^{-(2-\theta)/3}k^{-(2-\theta)/3} + \Theta_k\nu_k^{-1}, \end{aligned}$$

by Pólya's Inequality.

We have that $\Theta_k = \#\{(i_1, i_2, i_3) \in \mathbb{N}^3 : i_1^2 + i_2^2 + i_3^2 = \frac{\nu_k}{\pi^2}\}$ is the number of integer lattice points in the first octant that lie on the sphere in $\mathbb{R}^3$ which is centred at $(0, 0, 0)$ and has radius $\frac{\nu_k^{1/2}}{\pi}$. It is well known that $\#\{(x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1^2 + x_2^2 + x_3^2 = d\} = O(d^{\frac{1}{2}+o(1)})$.

The following routine proof was communicated by T. Wooley. Let $n = d - x_3^2$. Now $|x_3| \le d^{1/2}$, so for $x_3 \in [-d^{1/2}, d^{1/2}] \cap \mathbb{Z}$, there are at most $2d^{1/2} + 1$ possible values of n. If $n = 0$, then $x_1^2 + x_2^2 = 0$ has one solution $(0, 0) \in \mathbb{Z}^2$. Suppose that $n \ne 0$. Let $R(n)$ denote the number of pairs $(x_1, x_2) \in \mathbb{Z}^2$ such that $x_1^2 + x_2^2 = n$. Then

$$\#\{(x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1^2 + x_2^2 + x_3^2 = d\} \le 1 + \sum_{|z| \le d^{1/2}} R(d - z^2).$$

By Corollary 3.23 of [18], we have that

$$R(n) = 4 \sum_{d|n,\, d>0,\, d \text{ odd}} \left(\frac{-1}{d}\right),$$

where the sum is taken over all positive, odd divisors of n and $(\frac{-1}{d})$ is the quadratic residue symbol. Thus $R(n) \le 4D(n)$, where $D(n)$ denotes the number of positive divisors of n. By Theorem 8.31 of [18], for every $\epsilon > 0$, there exists n_ϵ such that for $n > n_\epsilon$,

$$D(n) < n^{(1+\epsilon)\log 2/\log\log n},$$

which implies that $D(n) = O(n^\epsilon)$. Therefore we obtain that

$$\#\{(x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1^2 + x_2^2 + x_3^2 = d\} \le 1 + O\Big(\sum_{|z| \le d^{1/2}} (d - z^2)^\epsilon\Big) \le 1 + O(d^{1/2+\epsilon}).$$

So $\Theta_k = O(\nu_k^{\frac{1}{2}+o(1)})$ and $\Theta_k \nu_k^{-1} = O(\nu_k^{-\frac{1}{2}+o(1)}) = O(k^{-\frac{1}{3}+o(1)})$. Thus we obtain

$$\frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k} \le \frac{3}{8\pi} + O(k^{-(2-\beta)/3}). \tag{5.10}$$

So by (5.6) and (5.10), we deduce that

$$\frac{1}{a_{1,k}^*} + \frac{1}{a_{2,k}^*} + \frac{1}{a_{3,k}^*} \le 3 + O(k^{-(2-\beta)/3}),\ k \to \infty. \tag{5.11}$$

Furthermore, by the Arithmetic Mean – Geometric Mean Inequality applied to $\frac{1}{a_{1,k}^*} + \frac{1}{a_{2,k}^*}$, we have by (5.11) that

$$2(a_{3,k}^*)^{1/2} + \frac{1}{a_{3,k}^*} \le 3 + O(k^{-(2-\beta)/3}),\ k \to \infty.$$

Let $a_{3,k}^* = 1 + \delta_k$ where $\delta_k > 0$. Then

$$2(1 + \delta_k)^{3/2} + 1 \le 3 + 3\delta_k + O(k^{-(2-\beta)/3}),\ k \to \infty.$$

Since $a_{3,k}^* \le 319$, $\delta_k \le 399$. Hence $(1 + \delta_k)^{3/2} \ge 1 + \frac{3}{2}\delta_k + \frac{3}{160}\delta_k^2$ for $0 < \delta_k \le 399$, we deduce that $\delta_k \le O(k^{-(2-\beta)/6})$, $k \to \infty$. As this estimate is independent of the subsequence $\big(R_{a_{1,k}^*, a_{2,k}^*, a_{3,k}^*}\big)_k$ we arrive at the conclusion of Theorem 1.1(ii).

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